UDC 539.3

ON EXISTENCE THEOREMS IN LINEAR SHELL THEORY

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A generalization of the Korn inequality which permits reduction of the proof of solvability of the problem of total shell energy minimization in some class of admissible displacements to the verification of some algebraic condition which the strains must satisfy, and to the proof of existence theorems for the solution (or to the verification of the equilibrium conditions). Existence theorems are proved by the scheme mentioned in the Novozhilov-Bolabukh shell theory [1] and in the Reissner theory [2, 3].

1. Let Ω be the domain of the variables $\mathbf{x} = (x_1, \ldots, x_n)$, and $\mathbf{u} = (u_1, \ldots, u_m)$ a vector function, let us say that $\mathbf{u} \in W_2^1(\Omega)$ if $u_i \in W_2^1(\Omega)$, $i = 1, \ldots, m$.

Let the linear first order differential operators with variable coefficients

$$\varepsilon_i^\circ(\mathbf{u}) = a_i^{jk} u_{j,k}, \qquad i = 1, \ldots, N \qquad (f_{,i} \equiv \partial f / \partial x_i)$$

$$\varepsilon_i(\mathbf{u}) = \varepsilon_i^\circ(\mathbf{u}) + b_i^j u_j, \qquad i = 1, \ldots, N$$

be given. We pose the question: Under what conditions on the operators $\varepsilon_i(\mathbf{u})$ for any vector function $\mathbf{u} \in W_2^1(\Omega)$ is the inequality generalizing the Korn inequality [4, 5] (see [5] in the References)

$$\|\mathbf{u}\|_{W^1_2(\Omega)} \leqslant c_1 \Big(\sum_{i=1}^{N} \|\varepsilon_i(\mathbf{u})\|^2_{L_2(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Omega)} \Big)^{1/2} \tag{1.1}$$

valid.

Theorem 1. Let Ω be such that its closure Ω^c is mapped holomorphically on some cube or sphere by using the mapping $T(\mathbf{x})$ of the class $C^3(\Omega^c)$ such that the Jacobian $|T'|$ has the positive constant c_T as lower bound. Let $a_i^{jk} \in C^2(\Omega^c)$, $b_i^j \in C(\Omega^c)$. Forming all possible first derivatives of the operators $\varepsilon_i^\circ(\mathbf{u})$ and extracting terms containing the second derivatives of the functions u_j, we obtain the differential expressions

$$\varepsilon_{ip} \equiv a_i^{jk} u_{j,kp}, \qquad f_{,ij} \equiv \partial^2 f / \partial x_i \partial x_j$$

It is sufficient for the validity of (1.1) that the following algebraic condition be satisfied: find functions $M_{lts}^{ip} \in C^1(\Omega^c)$ such that the identities

$$u_{l,ts} = M_{lts}^{ip} \varepsilon_{ip}(\mathbf{u}) \equiv M_{lts}^{ip} a_i^{jk} u_{j,kp} \tag{1.2}$$

hold. In other words, any second derivative of the functions u_j can be expressed in terms of a linear combination of differential expressions $\varepsilon_{ip}(\mathbf{u})$. The constant c_1 in (1.1) depends on the norm of the functions a_i^{jk}, M_{lts}^{ip}, b_i^j, respectively, in $C^2(\Omega^c)$, $C^1(\Omega^c)$, $C(\Omega^c)$, the norms of the mapping T in $C^3(\Omega^c)$, the constant c_T and the dimensions of Ω (c_1 increases as the dimensions decrease).

The assertion evidently follows form Theorem 1.

Theorem 2. Let the domain Ω be such that its closure is

$$\Omega^c = \Omega_1^c \cup \ldots \cup \Omega_k^c, \quad \Omega_i \cap \Omega_j = \Lambda, \quad i \neq j$$

and the conditions of Theorem 1 are satisfied for each domain Ω_i. Then the inequality (1.1) holds, where the constant c_1 in (1.1) is the maximum of the constants for the domains Ω_i.

Proof of Theorem 1. We introduce the notation: $D(\Omega)$ is the space of the fundamental functions, $D'(\Omega)$ is distribution space, $W_2^{1,0}(\Omega)$ is the space of functions belonging to $W_2^1(\Omega)$ and equal to zero on the boundary Ω, $W^{-1}(\Omega)$ is the space dual to $W_2^{1,0}(\Omega)$, $W^{-1}(\Omega) \subset D'(\Omega)$. If $f \in D'(\Omega)$, $\varphi \in D(\Omega)$, the value of f in the function φ will be denoted by $(f, \varphi)_\Omega$.

We introduce the Hilbert space $Y(\Omega)$ consisting of the distributions $f \in W^{-1}(\Omega)$ such that $f_{,i} \in W^{-1}(\Omega)$, $i = 1, \ldots, n$, and we assume

$$\|f\|_{Y(\Omega)} \equiv \left(\|f\|^2_{W^{-1}(\Omega)} + \sum_{i=1}^{n} \|f_{,i}\|^2_{W^{-1}(\Omega)}\right)^{1/2} \tag{1.3}$$

Lemma 1. $L_2(\Omega)$ is imbedded continuously in $W^{-1}(\Omega)$ and in $Y(\Omega)$, where

$$\|f_{,i}\|_{W^{-1}(\Omega)} \leqslant \|f\|_{L_2(\Omega)}, \qquad i = 1, \ldots, n$$

$$\|f\|_{W^{-1}(\Omega)} \leqslant \|f\|_{L_2(\Omega)}, \quad \|f\|_{Y(\Omega)} \leqslant c_2 \|f\|_{L_2(\Omega)}, \quad c_2 = n + 1$$

Lemma 2 (fundamental). The space $Y(\Omega)$ is imbedded continuously in $L_2(\Omega)$, i.e. if the distribution $f \in Y(\Omega)$, then $f \in L_2(\Omega)$ and

$$\|f\|_{L_2(\Omega)} \leqslant c_3 \|f\|_{Y(\Omega)} \tag{1.4}$$

Proof. Let T be the mapping transferring Ω into a cube (or sphere) G. We construct the mapping P of $Y(\Omega)$ into $Y(G)$

$$(Pf, \varphi)_G \equiv (f, \varphi T)_\Omega$$

(φT is the superposition of φ and T). It can be verified that P is a linear homeomorphism between $Y(\Omega)$ and $Y(G)$, where

$$\|Pf\|_{Y(G)} \leqslant c_4 \|f\|_{Y(\Omega)}, \quad \|P^{-1}g\|_{Y(\Omega)} \leqslant c_4 \|g\|_{Y(G)} \tag{1.5}$$

Here c_4 depends on the norm of T in $C^3(\Omega^c)$ and the constant c_T. It can be verified that P is also a linear homeomorphism between $L_2(\Omega)$ and $L_2(G)$, and if $f \in L_2(\Omega)$, $g \in L_2(G)$, then

$$\|Pf\|_{L_2(G)} \leqslant c_5 \|f\|_{L_2(\Omega)}, \quad \|P^{-1}g\|_{L_2(\Omega)} \leqslant c_5 \|g\|_{L_2(G)} \tag{1.6}$$

Here c_5 depends on the norm of T in $C^1(\Omega^c)$ and the constant c_T.

Lemma 2 has been proved in [4] for an arbitrary domain with smooth boundary (and therefore for a sphere G also). A slight addition permits the proof of Lemma 2 for the cube G, i.e. if $g \in Y(G)$, then $g \in L_2(G)$ and $\|g\|_{L_2(\Omega)} \leqslant c_6 \|g\|_{Y(G)}$, and (1.4) with the constant $c_3 = c_4 c_5 c_6$ follows from (1.5), (1.6).

By condition (1.2)

$$u_{l,ts} = M^{ip}_{lts} \varepsilon_{ip}(\mathbf{u}) \equiv M^{ip}_{lts} [\mathring{\varepsilon}_{i,p}(\mathbf{u}) - a^{jk}_{i,p} u_{j,k}] \tag{1.7}$$

From Lemmas 1 and 2, (1.3), (1.7) there follows that ($I \equiv \|\mathbf{u}\|^2_{L_2(\Omega)}$)

$$\|\mathbf{u}\|^2_{W_2^1(\Omega)} \leqslant I + c_3^2 \sum_{l,t} \|u_{l,t}\|^2_{Y(\Omega)} = I + c_3^2 \sum_{l,t} \|u_{l,t}\|^2_{W^{-1}(\Omega)} + \tag{1.8}$$

$$c_3^2 \sum_{l,t,s} \|u_{l,ts}\|^2_{W^{-1}(\Omega)} \leqslant (1 + c_3^2 n) I + c_3^2 \sum_{l,t,s} \|M^{ip}_{lts} \mathring{\varepsilon}_{i,p}\|^2_{W^{-1}(\Omega)} + c_3^2 \sum_{l,t,s} \|M^{ip}_{lts} a^{jk}_{i,p} u_{j,k}\|^2_{W^{-1}(\Omega)}$$

The inequality (1.1) results from (1.8) and the following assertion: let $f \in L_2(\Omega)$, $g \in C^1(\Omega)$, then $gf_{,i} \in W^{-1}(\Omega)$, and

$$\|gf_{,i}\|_{W^{-1}(\Omega)} \leqslant \|g\|_{C^1(\Omega^c)} \|f\|_{L_2(\Omega)}$$

2. Let the shell middle surface S be given by the equation $\mathbf{r} = \mathbf{r}(\mathbf{x})$ which homeomorphically maps S onto the domain Ω of the variables $\mathbf{x}=(x_1, x_2)$ satisfying the condition of Theorem 2, the Lamé coefficients by $A_1, A_2 \in C^2(\Omega^c)$, $A_1, A_2 \geqslant m > 0$, $m = \text{const}$, the curvatures by $R_1^{-1}, R_2^{-1} \in C^1(\Omega^c)$.

Let us investigate the solvability of the Novozhilov-Bolabukh shell equations [1]. Let us introduce the space of displacement fields and the known functions

$$H_1(\Omega) = \{\mathbf{U} \mid \mathbf{U} = (\mathbf{u}, w),\ \mathbf{u} = (u_1, u_2),\ \mathbf{u} \in W_2^1(\Omega),\ w \in W_2^2(\Omega)\}$$

$$\|\mathbf{U}\|_{H_1(\Omega)} \equiv (\|\mathbf{u}\|^2_{W_2^1(\Omega)} + \|w\|^2_{W_2^2(\Omega)})^{1/2} \tag{2.1}$$

$$\vartheta_1 = -A_1^{-1}w_{,1} + R_1^{-1}u_1, \qquad \vartheta_2 = -A_2^{-1}w_{,2} + R_2^{-1}u_2$$

$$\omega_1 = A_1^{-1}u_{2,1} - A_{1,2}(A_1A_2)^{-1}u_1, \qquad \omega_2 = A_2^{-1}u_{1,2} - A_{2,1}(A_1A_2)^{-1}u_2 \tag{2.2}$$

$$\tau_1 = A_1^{-1}\vartheta_{2,1} - A_{1,2}(A_1A_2)^{-1}\vartheta_2, \quad \tau_2 = A_2^{-1}\vartheta_{1,2} - A_{2,1}(A_1A_2)^{-1}\vartheta_1 \tag{2.3}$$

Let ε denote the set of strains

$$\varepsilon_1 = A_1^{-1}u_{1,1} + (A_1A_2)^{-1}A_{1,2}u_2 + R_1^{-1}w, \quad \varepsilon_2 = A_2^{-1}u_{2,2} + (A_1A_2)^{-1}A_{2,1}u_1 + R_2^{-1}w \tag{2.4}$$

$$\varkappa_1 = A_1^{-1}\vartheta_{1,1} + (A_1A_2)^{-1}A_{1,2}\vartheta_2, \qquad \varkappa_2 = A_2^{-1}\vartheta_{2,2} + (A_1A_2)^{-1}A_{2,1}\vartheta_1 \tag{2.5}$$

$$\omega = \omega_1 + \omega_2,\ \tau = 2^{-1}(\tau_1 + \tau_2 + R_1^{-1}\omega_2 + R_2^{-1}\omega_1)$$

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \omega, \varkappa_1, \varkappa_2, \tau),\ \|\varepsilon\|_{L_2(\Omega)} \equiv \left[\int_\Omega (\varepsilon_1^2 + \varepsilon_2^2 + \omega^2 + \varkappa_1^2 + \varkappa_2^2 + \tau^2)\, d\mathbf{x}\right]^{1/2}$$

Theorem 3. For any field $\mathbf{U} \in H_1(\Omega)$ the inequality

$$\|\mathbf{U}\|_{H_1(\Omega)} \leqslant c_7 (\|\varepsilon\|^2_{L_2(\Omega)} + \|\mathbf{u}\|^2_{L_2(\Omega)} + \|w\|^2_{W_2^1(\Omega)})^{1/2} \tag{2.6}$$

holds.

Proof. Forming all possible first derivatives of the strains ε_1, ε_2, ω and extracting terms containing the second derivatives of the functions u_1, u_2, we obtain differential expressions satisfying condition (1.2)

$$\varepsilon_{11} \equiv A_1^{-1}u_{1,11},\ \varepsilon_{12} \equiv A_1^{-1}u_{1,12},\ \varepsilon_{21} \equiv A_2^{-1}u_{2,21},\ \varepsilon_{22} \equiv A_2^{-1}u_{2,22}$$

$$\omega_{11} \equiv A_1^{-1}u_{2,11} + A_2^{-1}u_{1,21},\quad \omega_{22} \equiv A_1^{-1}u_{2,12} + A_2^{-1}u_{1,22}$$

In fact, $u_{1,11} = A_1\varepsilon_{11}$, $u_{1,12} = A_1\varepsilon_{12}$, $u_{1,22} = A_2\omega_{22} - A_1^{-1}A_2^2\varepsilon_{21}$, the derivatives of u_2 are expressed analogously, hence, the inequality

$$\|\mathbf{u}\|_{W_2^1(\Omega)} \leqslant c_8\left[\int_\Omega (\varepsilon_1^2 + \varepsilon_2^2 + w^2)\, d\mathbf{x} + \|\mathbf{u}\|^2_{L_2(\Omega)} + \|w\|^2_{L_2(\Omega)}\right]^{1/2} \tag{2.7}$$

follows from Theorem 2.

Since the strains $\varkappa_1$, $\varkappa_2$, τ contain the senior terms $-A_1^{-2}w_{,11}$, $-A_2^{-2}w_{,22}$, $-(A_1A_2)^{-1}w_{,12}$, respectively, we obtain (2.6) from (2.7).

Let us introduce the total energy functional $\Phi_1(\mathbf{U}) = E_1(\mathbf{U}) - L_1(\mathbf{U})$, where $E_1(\mathbf{U})$ is the strain energy [1], and $L_1(\mathbf{U})$ is the work of the external forces (a linear functional continuous in $H_1(\Omega)$).

Let $H_1^\circ(\Omega)$ denote the subspace of $H_1(\Omega)$ consisting of fields $\mathbf{U}$ such that $\varepsilon=0$. It is known [6] that $H_1^\circ(\Omega)$ consists of displacement fields of the shell as a rigid whole.

Theorem 4. In order for the problem of minimizing the functional $\Phi_1(U)$ to have a solution in the space of admissible displacement fields $H_1^*(\Omega) \subset H_2(\Omega)$, it is necessary and sufficient that the equilibrium conditions be satisfied: for any field $U \in R_1^\circ \times (\Omega) \equiv H_1^\circ(\Omega) \cap H_1^*(\Omega)$, $L_1(U) = 0$; the solution is determined to the accuracy of an arbitrary field from $R_1^\circ(\Omega)$. In particular, if $R_1^\circ(\Omega) = 0$ (i.e., the boundary conditions prevent the displacement of the shell as a rigid whole), the equilibrium condition is satisfied trivially, and the solution exists and is unique.

Proof. We form the factor-space $H(\Omega) = H_1^*(\Omega) / R_1^\circ(\Omega)$ and we define the norm in $H(\Omega)$ as follows

$$\| U \|_{H(\Omega)} \equiv [E_1(U)]^{1/2}$$

Considering the opposite, and using (2.6), as well as the inequality $E_1(U) \geqslant c_9 \| \varepsilon \|^2_{L_2(\Omega)}$, it can be shown that the functional $L_1(U)$ is continuous in $H(\Omega)$ from which the assertion of the theorem follows [7].

3. Let us investigate the solvability of the Reissner shell equations [2, 3]. We introduce the space of displacement fields

$$H_2(\Omega) = \{V \mid V = (u_1, u_2, w, \vartheta_1, \vartheta_2),\ V \in W_2^1(\Omega)\}$$

The notation of (2.2), (2.3) is retained, and (2.4), (2.5) specify the strains ε_1, ε_2, $\varkappa_1$, $\varkappa_2$

$$\tau_1^\circ \equiv \tau_1 + R_1^{-1}\omega_2, \qquad \tau_2^\circ \equiv \tau_2 + R_2^{-1}\omega_1$$

$$\varepsilon_{12} = \frac{\omega_1 + \omega_2}{2} + \frac{h^2}{48}(R_2^{-1} - R_1^{-1})\left[\tau_1^\circ - \tau_2^\circ + \frac{\omega_1 + \omega_2}{2}(R_2^{-1} - R_1^{-1})\right]$$

$$\varkappa_{12} = \frac{\tau_1^\circ + \tau_2^\circ}{2} - \frac{1}{4}(R_2^{-1} + R_1^{-1})(\omega_1 + \omega_2)$$

$$\gamma_1 = A_1^{-1}w_{,1} - R_1^{-1}u_1 + \vartheta_1, \qquad \gamma_2 = A_2^{-1}w_{,2} - R_2^{-1}u_2 + \vartheta_2$$

$$\boldsymbol{\varepsilon}_R \equiv (\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \varkappa_1, \varkappa_2, \varkappa_{12}, \gamma_1, \gamma_2)$$

$$\| \boldsymbol{\varepsilon}_R \|_{L_2(\Omega)} \equiv \left[\int_\Omega (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_{12}^2 + \varkappa_1^2 + \varkappa_2^2 + \varkappa_{12}^2 + \gamma_1^2 + \gamma_2^2)\,d\mathbf{x}\right]^{1/2}$$

Here h is the shell thickness. It can be proved that if

$$\max\ \{hR_1^{-1}, hR_2^{-1}\} \leqslant 1 - \nu \tag{3.1}$$

(ν is the Poisson's ratio), then the strain energy of a Reissner shell is a positive definite quadratic form of the strain ε_R.

Theorem 5. For any field $V \in H_2(\Omega)$ the inequality

$$\| V \|_{W_2^1(\Omega)} \leqslant c_{10} (\| \boldsymbol{\varepsilon}_R \|^2_{L_2(\Omega)} + \| V \|^2_{L_2(\Omega)})^{1/2} \tag{3.2}$$

is valid.

Proof. Let us differentiate the strains ε_1, ε_2, $\varkappa_1$, $\varkappa_2$, γ_1, γ_2 and extract the terms containing the second derivatives of the functions u_1, u_2, ϑ_1, ϑ_2, w. All the second derivatives of the functions u_1, u_2, ϑ_1, ϑ_2, w except $u_{1,22}$, $u_{2,11}$, $\vartheta_{1,22}$, $\vartheta_{2,11}$ can be expressed in terms of the differential expressions obtained in this manner.

Differentiating ε_{12} and $\varkappa_{12}$ with respect to x_2 and transposing terms containing $u_{1,22}$ and $\vartheta_{1,22}$ to the left, we obtain the system

$$\left\{\frac{h^2}{48}(R_2^{-1} - R_1^{-1})R_1^{-1}A_2^{-1} + \frac{1}{2}A_2^{-1}\left[1 + \frac{h^2}{48}(R_2^{-1} - R_1^{-1})^2\right]\right\} u_{1,22} - \tag{3.3}$$

$$\frac{h^3}{48}(R_2^{-1}-R_1^{-1})A_2^{-1}\vartheta_{1,22}=b_1\,, \qquad {}^1/_4(R_1^{-1}-R_2^{-1})A_2^{-1}u_{1,22}+{}^1/_2A_2^{-1}\vartheta_{1,22}=b_2$$

to determine them. The right sides b_1, b_2 in (3.3) are composed of the derivatives already found. The system (3.3) is solvable under the condition (3.1), conditions (1.2) are satisfied, and (3.2) follows from Theorem 2.

Let $H_2^\circ(\Omega)$ denote the subspace of $H_2(\Omega)$ which consists of fields V such that $\varepsilon_R = 0$. Then the functions ϑ_1, ϑ_2 are expressed in terms of u_1, u_2, w by means of (2.1), hence $H_2^\circ(\Omega)$ has the form

$$H_2^\circ(\Omega) = \{V \mid V = (u_1, u_2, w, \vartheta_1, \vartheta_2),\ (u_1, u_2, w) \in H_1^\circ(\Omega),\ \vartheta_i = -A_i^{-1}w_{,i} + R_i^{-1}u_i,\ i = 1,2\}$$

An existence theorm holds for the solution which is completely analogous to Theorem 4. Other shell equations, [8] say, can also be investigated by the same scheme.

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